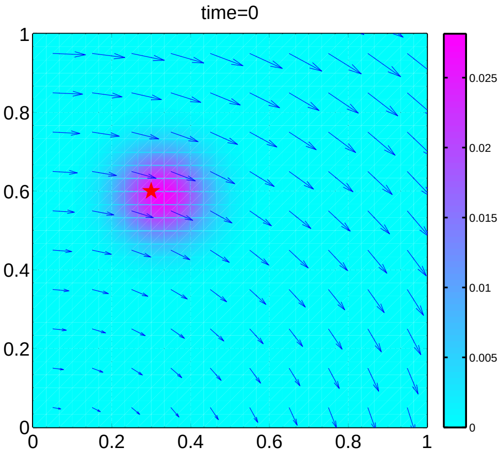
Maximin efficient sensor location for estimation of subsets of parameters

Maciej Patan **Dariusz Uciński**

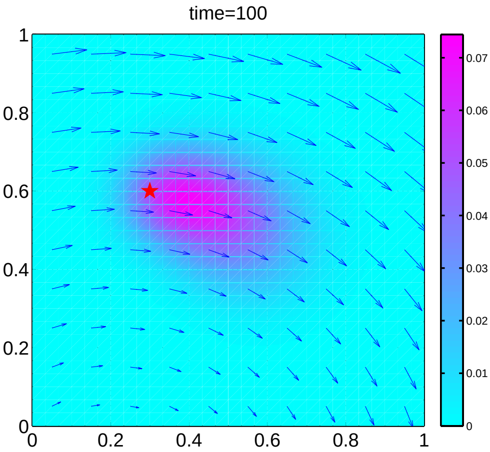
Institute of Control and Computation Eng., University of Zielona Góra, Poland

14-th Int. Workshop *Model-Oriented Design and Analysis*
mODa 14, Drongen Abbey, Belgium, June 16, 2026

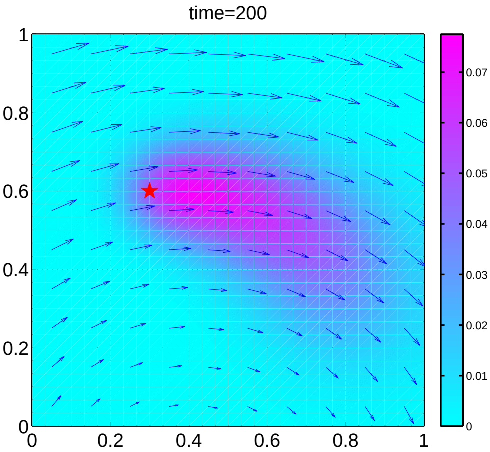
Example: Air pollution monitoring



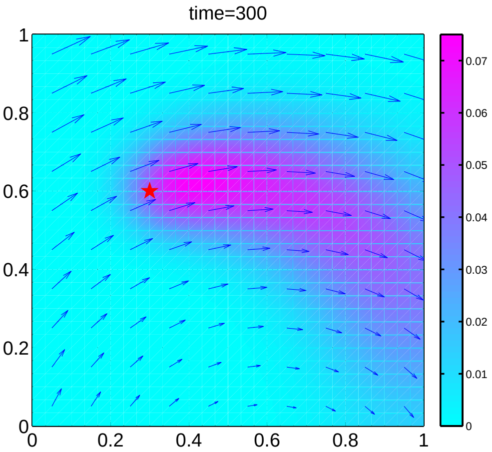
Example: Air pollution monitoring



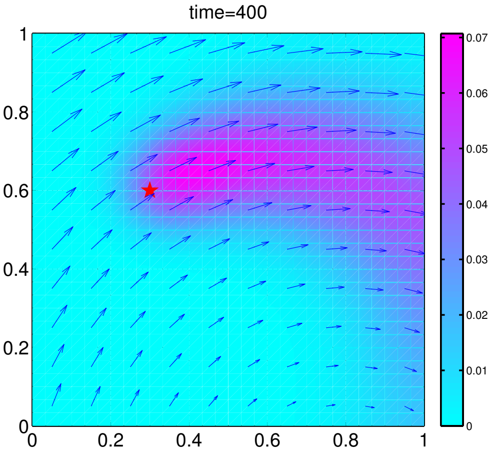
Example: Air pollution monitoring



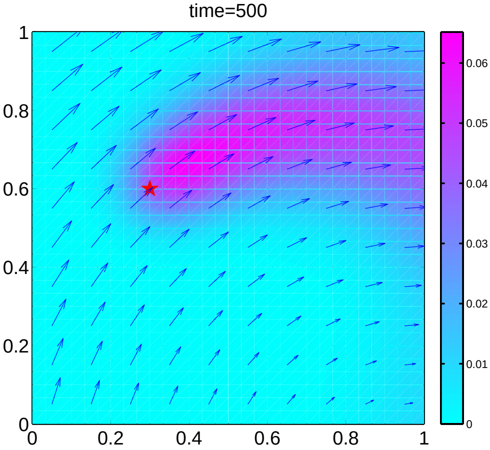
Example: Air pollution monitoring



Example: Air pollution monitoring



Example: Air pollution monitoring



Example: Air pollution

Consider domain $\Omega \subset \mathbb{R}^2$ with sufficiently smooth boundary Γ.

Advection-diffusion equation

$$\frac{\partial y}{\partial t}(\boldsymbol{x}, t) = \nabla \cdot [\kappa(\boldsymbol{x}, \boldsymbol{\theta}) \nabla y(\boldsymbol{x}, t)] + \boldsymbol{\nu}(\boldsymbol{x}) \cdot \nabla y(\boldsymbol{x}, t) + f(\boldsymbol{x}, t, \boldsymbol{\theta})$$

$$y(\boldsymbol{x}, 0) = y_0(\boldsymbol{x}), \quad y(\boldsymbol{x}, t)\Big|_{\partial\Omega} = 0$$

Notation:

- $y(\boldsymbol{x}, t)$ – contaminant concentrat. at point $\boldsymbol{x}$ and time t
- $\kappa(\boldsymbol{x}, \boldsymbol{\theta})$ – diffusivity coefficient, parameterized by $\boldsymbol{\theta}$
- $f(\boldsymbol{x}, t, \boldsymbol{\theta})$ – source term
- $\boldsymbol{\nu}(\boldsymbol{x}) = (\nu_1(\boldsymbol{x}), \nu_2(\boldsymbol{x}))$ – wind velocity field

System description

Consider a spatial domain $\Omega \subset \mathbb{R}^2$ and the PDE

$$\frac{\partial y}{\partial t} = \mathcal{F}\left(\boldsymbol{x}, t, y, \boldsymbol{\theta}\right), \quad \boldsymbol{x} \in \Omega,\ t \in T$$

subject to the appropriate initial and boundary conditions.

Notation:

- $\boldsymbol{x}$ – spatial variable, t – time, $T = [0, t_f]$;
- $y = y(\boldsymbol{x}, t)$ – state variable;
- $\mathcal{F}$ – known operator including spatial derivatives;
- $\boldsymbol{\theta} \in \mathbb{R}^m$ – vector of *unknown* (constant) parameters.

Observations

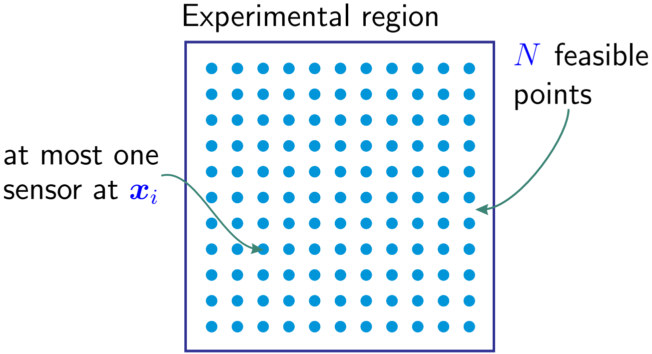
Measurements by pointwise stationary sensors

$$z_j^\ell = y(\boldsymbol{x}_j, t_\ell; \boldsymbol{\theta}) + \varepsilon(\boldsymbol{x}_j, t_\ell), \quad \begin{cases} j = 1, \dots, n \\ \ell = 1, \dots, L \end{cases}$$

z_j^ℓ	scalar response
$\theta_1, \dots, \theta_m$	unknown parameters, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m)^\top$
ε	uncorrelated random disturbance (zero mean and constant variance)

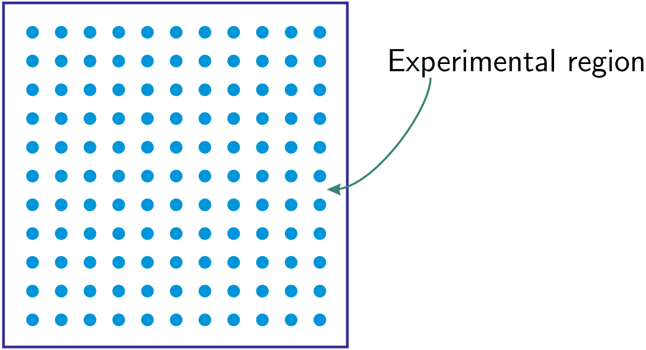
Observations

Given n sensors and a finite set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ (here $N > n$) of candidate points for their location, consider an 'optimal' selection of these observation points.



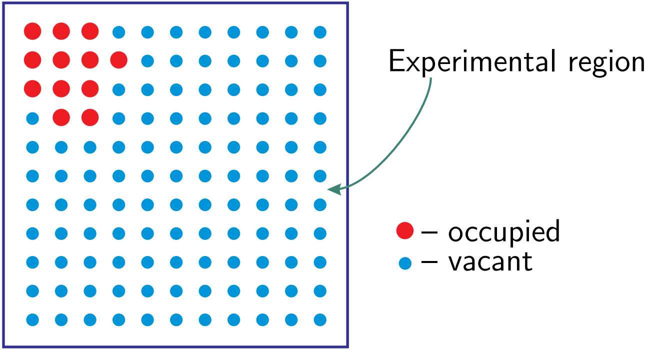
Observations

Combinatorial problem: Which n-element subset to select?



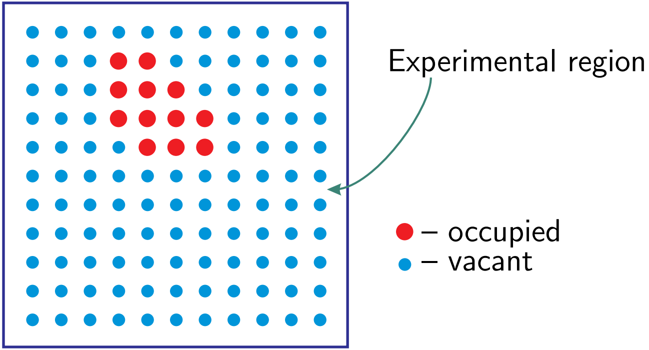
Observations

Combinatorial problem: Which n-element subset to select?



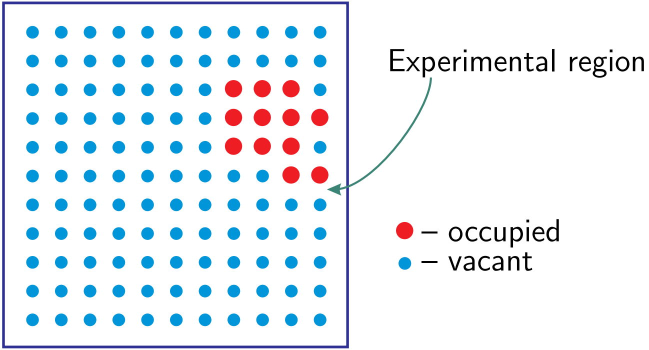
Observations

Combinatorial problem: Which n-element subset to select?



Observations

Combinatorial problem: Which n-element subset to select?



Binary decision variables

Given candidate points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_N$, introduce binary flags (indicator variables) $\boldsymbol{v} = (v_1, \dots, v_N)$ satisfying

$$v_i = \begin{cases} 1 & \text{if a sensor is located at } \boldsymbol{x}_i \\ 0 & \text{otherwise} \end{cases}$$

Thus looking for a best n-element subset of N design points amounts to choosing the best element of the feasible set of binary weights

$$\mathcal{V}_{\mathsf{bin}} = \left\{ \boldsymbol{v} \in \{0, 1\}^N : \boldsymbol{v}^\top \mathbf{1} = n \right\}$$

Notation: $\mathbf{1}$ – vector of ones

Optimal measurement problem

Determine $\boldsymbol{v} \in \mathcal{V}_{\mathsf{bin}}$ so as to maximize the identification accuracy

Optimal measurement problem

Determine $\boldsymbol{v} \in \mathcal{V}_{\mathsf{bin}}$ so as to maximize the identification accuracy. $\leftarrow$ What is it?

Optimal measurement problem

Determine $\boldsymbol{v} \in \mathcal{V}_{\text{bin}}$ so as to maximize the identification accuracy. $\leftarrow$ What is it?

Cramér-Rao inequality

$$\operatorname{cov} \hat{\boldsymbol{\theta}} = E\Big\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^\top\Big\} \succeq \boldsymbol{M}^{-1}$$

We have $\operatorname{cov} \hat{\boldsymbol{\theta}} = \boldsymbol{M}^{-1}$ provided that the estimator is efficient. But what is $\boldsymbol{M}$?

Fisher Information Matrix (FIM)

Up to a constant multiplier, we have

$$\boldsymbol{M}(\boldsymbol{v}) = \boldsymbol{M}_0 + \sum_{i=1}^{N} v_i \boldsymbol{M}_i, \qquad \boldsymbol{M}_i = \sum_{\ell=1}^{L} \boldsymbol{g}(\boldsymbol{x}_i, t_\ell)\boldsymbol{g}^\top(\boldsymbol{x}_i, t_\ell)$$

where $\boldsymbol{g}(\boldsymbol{x}, t) = \left(\dfrac{\partial y(\boldsymbol{x}, t; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^\top_{\boldsymbol{\theta}=\boldsymbol{\theta}^0}$ is the vector of sensitivity coefficients, $\boldsymbol{\theta}^0$ stands for a preliminary estimate of $\boldsymbol{\theta}$.

Calculation of the sensitivity vector $\boldsymbol{g}$

- Finite-difference method
- Sensitivity-equation method (direct approach)
- Variational method (indirect approach)

(Relaxed) problem of best sensor selection

Find $v_1, \dots, v_N$ so as to maximize the Φ-optimum design criterion

$$\mathcal{P}_\Phi(\boldsymbol{v}) = \Phi\Big(\boldsymbol{M}_0 + \sum_{i=1}^{N} v_i \boldsymbol{M}_i\Big)$$

subject to

$$\sum_{i=1}^{N} v_i = n$$

$$v_i \in \big\{0, 1\big\}, \quad i = 1, \dots, N$$

Φ is assumed to be isotonic, positive homogeneous, upper semicontinuous and concave, e.g., for D-optimality

$$\Phi_{\mathsf{D}}(\boldsymbol{M}) = \det{}^{1/m}(\boldsymbol{M})$$

and its maximization yields the minimal volume of the confidence ellipsoid.

(Relaxed) problem of best sensor selection

Find $v_1, \dots, v_N$ so as to maximize the Φ-optimum design criterion

$$\mathcal{P}_\Phi(\boldsymbol{v}) = \Phi\Big(\boldsymbol{M}_0 + \sum_{i=1}^{N} v_i \boldsymbol{M}_i\Big)$$

subject to

$$\sum_{i=1}^{N} v_i = n$$

$$v_i \in [0, 1], \quad i = 1, \dots, N$$

Φ is assumed to be isotonic, positive homogeneous, upper semicontinuous and concave, e.g., for D-optimality

$$\Phi_{\mathsf{D}}(\boldsymbol{M}) = \det{}^{1/m}(\boldsymbol{M})$$

and its maximization yields the minimal volume of the confidence ellipsoid.

(Relaxed) problem of best sensor selection

Find $v_1, \dots, v_N$ so as to maximize the Φ-optimum design criterion

$$\mathcal{P}_\Phi(\boldsymbol{v}) = \Phi\Big(\boldsymbol{M}_0 + \sum_{i=1}^{N} v_i \boldsymbol{M}_i\Big)$$

subject to

$$\sum_{i=1}^{N} v_i = n$$

$$v_i \in [0, 1], \quad i = 1, \dots, N$$

Φ is assumed to be isotonic, positive homogeneous, upper semicontinuous and concave, e.g., for E-optimality

$$\Phi_{\mathsf{E}}(\boldsymbol{M}) = \lambda_{\min}(\boldsymbol{M})$$

and its maximization yields the minimal length of the largest axis of the confidence ellipsoid.

(Relaxed) problem of best sensor selection

Find $v_1, \dots, v_N$ so as to maximize the Φ-optimum design criterion

$$\mathcal{P}_\Phi(\boldsymbol{v}) = \Phi\Big(\boldsymbol{M}_0 + \sum_{i=1}^{N} v_i \boldsymbol{M}_i\Big)$$

subject to

$$\sum_{i=1}^{N} v_i = n$$

$$v_i \in [0, 1], \quad i = 1, \dots, N$$

Φ is assumed to be isotonic, positive homogeneous, upper semicontinuous and concave, e.g., for A-optimality

$$\Phi_{\mathsf{A}}(\boldsymbol{M}) = \big(\operatorname{trace}(\boldsymbol{M}^{-1})\big)^{-1}$$

and its maximization yields the minimal sum of the squared lengths of the axes of the confidence ellipsoid.

Which optimality criterion to choose?

Different optimality criteria focus on different aspects of the FIM. As a result, they may lead to radically different sensor locations and an optimal solution for one criterion may prove poor for another criterion of interest.

Therefore, practitioners can be interested in designs representing the best compromise between a sufficiently broad class of optimum design criteria.

Orthogonally invariant information criteria (Harman, 2004)

A criterion Φ is called orthogonally invariant if

$$\Phi(\boldsymbol{U}\boldsymbol{M}\boldsymbol{U}^\top) = \Phi(\boldsymbol{M})$$

for any FIM and any orthogonal (i.e., satisfying $\boldsymbol{U}^\top\boldsymbol{U} = \boldsymbol{I}_m$) matrix $\boldsymbol{U} \in \mathbb{R}^{m\times m}$. Equivalently, $\Phi(\boldsymbol{M})$ only depends on the eigenvalues of $\boldsymbol{M}$.

The class $\mathbb{O}$ of orthogonally invariant design criteria is broad enough to include all 'alphabetical'criteria widely used in practice (D-, A-, E-optimal, and many others).

Maximin-efficient sensor location

The Φ-suboptimality of a sensor location $\boldsymbol{v}$ is quantified by its efficiency:

$$\operatorname{eff}(\boldsymbol{v} \mid \Phi) = \frac{\Phi(\boldsymbol{M}(\boldsymbol{v}))}{\Phi(\boldsymbol{M}(\boldsymbol{v}_\Phi^\star))}$$

where $\boldsymbol{v}_\Phi^\star$ is an Φ-optimum sensor location. Note that $0 \le \operatorname{eff}(\boldsymbol{v} \mid \Phi) \le 1 = \operatorname{eff}(\boldsymbol{v}_\Phi^\star \mid \Phi)$.

Maximin-efficient sensor location

The Φ-suboptimality of a sensor location $\boldsymbol{v}$ is quantified by its efficiency:

$$\mathrm{eff}(\boldsymbol{v} \mid \Phi) = \frac{\Phi(\boldsymbol{M}(\boldsymbol{v}))}{\Phi(\boldsymbol{M}(\boldsymbol{v}^\star_\Phi))}$$

where $\boldsymbol{v}^\star_\Phi$ is an Φ-optimum sensor location. Note that $0 \le \mathrm{eff}(\boldsymbol{v} \mid \Phi) \le 1 = \mathrm{eff}(\boldsymbol{v}^\star_\Phi \mid \Phi)$.

Problem

Find $\boldsymbol{v}^\star \in \mathcal{V}$ to maximize the $\mathbb{O}$-maximin-efficiency design criterion

$$P(\boldsymbol{v}) = \inf_{\Phi \in \mathbb{O}} \mathrm{eff}(\boldsymbol{v} \mid \Phi)$$

Thus, we are to maximize the worst efficiency within the class of orthogonall invariant criteria $\mathbb{O}$.

Powerful reframing

Define the E_k-optimality criterion

$$\Phi_{\mathsf{E}_k}(M) = \sum_{\ell=1}^{k} \lambda_\ell(\boldsymbol{M})$$

where $\lambda_1(\boldsymbol{M}) \leq \cdots \leq \lambda_m(\boldsymbol{M})$ denote the eigenvalues of $\boldsymbol{M}$ in increasing order.

Powerful reframing

Define the E_k-optimality criterion

$$\Phi_{\mathsf{E}_k}(M) = \sum_{\ell=1}^{k} \lambda_\ell(\boldsymbol{M})$$

where $\lambda_1(\boldsymbol{M}) \leq \cdots \leq \lambda_m(\boldsymbol{M})$ denote the eigenvalues of $\boldsymbol{M}$ in increasing order.

We have (Harman, 2004)

$$\inf_{\Phi \in \mathbb{O}} \operatorname{eff}(\boldsymbol{v} \mid \Phi) = \min_{k=1,\dots,m} \operatorname{eff}(\boldsymbol{v} \mid \Phi_{\mathsf{E}_k})$$

Hence the minimal efficiency w.r.t. the uncountable set of all orthogonally invariant criteria is simply the minimal efficiency w.r.t. the set of criteria Φ_{E_k} which has only m elements.

Reduction to semi-infinite programming (SIP)

For $\boldsymbol{M} \succ \mathbf{0}$ the following *minimum principle* holds:

$$\Phi_{\mathsf{E}_k}(\boldsymbol{M}) = \min_{\boldsymbol{Q} \in \mathcal{Q}_k} \operatorname{trace}(\boldsymbol{Q}^\top \boldsymbol{M} \boldsymbol{Q})$$

where

$$\mathcal{Q}_k = \left\{ \boldsymbol{Q} \in \mathbb{R}^{m \times k} : \boldsymbol{Q}^\top \boldsymbol{Q} = \boldsymbol{I}_k \right\}$$

Reduction to semi-infinite programming (SIP)

For $\boldsymbol{M} \succ \mathbf{0}$ the following *minimum principle* holds:

$$\Phi_{\mathsf{E}_k}(\boldsymbol{M}) = \min_{\boldsymbol{Q} \in \mathcal{Q}_k} \operatorname{trace}(\boldsymbol{Q}^\top \boldsymbol{M} \boldsymbol{Q})$$

where

$$\mathcal{Q}_k = \left\{ \boldsymbol{Q} \in \mathbb{R}^{m \times k} : \boldsymbol{Q}^\top \boldsymbol{Q} = \boldsymbol{I}_k \right\}$$

Problem

Find $\boldsymbol{v} \in \mathcal{V}$ to maximize

$$P(\boldsymbol{v}) = \min_{k=1,\dots,m} \left(\frac{1}{c_k} \min_{\boldsymbol{Q} \in \mathcal{Q}_k} \operatorname{trace}(\boldsymbol{Q}^\top \boldsymbol{M}(\boldsymbol{v}) \boldsymbol{Q}) \right)$$

where $c_k = \max_{\boldsymbol{v} \in \mathcal{V}} \Phi_{\mathsf{E}_k}(\boldsymbol{M}(\boldsymbol{v}))$.

Reduction to SIP (cont'd)

Problem

Find $(\boldsymbol{v}, \alpha) \in \mathcal{V} \times \mathbb{R}$ *so as to maximize* α *subject to the constraints*

$$\frac{1}{c_k} \operatorname{trace}(\boldsymbol{Q}_k^\top \boldsymbol{M}(\boldsymbol{v}) \boldsymbol{Q}_k) \geq \alpha, \quad \forall \boldsymbol{Q}_k \in \mathcal{Q}_k$$

for $k = 1, \ldots, m$.

Reduction to SIP (cont'd)

Problem

Find $(\boldsymbol{v}, \alpha) \in \mathcal{V} \times \mathbb{R}$ *so as to maximize* α *subject to the constraints*

$$\frac{1}{c_k}\operatorname{trace}(\boldsymbol{Q}_k^\top \boldsymbol{M}(\boldsymbol{v})\boldsymbol{Q}_k) \geq \alpha, \quad \forall \boldsymbol{Q}_k \in \mathcal{Q}_k$$

for $k = 1, \ldots, m$.

This problem is solved using a relaxation procedure (the method of outer approximations) of Shimizu and Aiyoshi (1980). It alternates between computing the eigenvalues of the current FIM along with their associated orthonormal eigenvectors, and solving a linear-programming problem with respect to the design weights $\boldsymbol{v}$, cf. (Burclová and Pázman, 2016; Uciński and Patan, 2021).

Problem to be attacked here

In many settings interest is in estimating a subset of s of the parameters as precisely as possible. Without loss of generality, we may write

$$\boldsymbol{\theta} = \left[\begin{array}{ccc|ccc} \theta_1 & \dots & \theta_s & \theta_{s+1} & \dots & \theta_m \end{array}\right]^\top = \left[\begin{array}{c|c} \boldsymbol{\theta}_{[1]}^\top & \boldsymbol{\theta}_{[2]}^\top \end{array}\right]^\top$$

where

- $\boldsymbol{\theta}_{[1]} \in \mathbb{R}^s$ is the vector of the parameters of interest
- $\boldsymbol{\theta}_{[2]} \in \mathbb{R}^{m-s}$ is the vector of nuisance parameters

Problem to be attacked here (cont'd)

How to modify $\mathbb{O}$-maximin-efficiency criterion and then numerically construct the appropriate optimum designs?

Problem to be attacked here (cont'd)

How to modify $\mathbb{O}$-maximin-efficiency criterion and then numerically construct the appropriate optimum designs?

Our idea draws on the analogous approach to alphabetical criteria, e.g., D_s-optimality. Anticipating what follows:

- Bad news: the appealing simple LP steps in the relaxation scheme can no longer be applied.
- Good news: they are replaced by convex optimization problems made affordable using generalized simplicial decomposition.

Partition of FIM

Assuming that $\boldsymbol{M}(\boldsymbol{v}) \succ \boldsymbol{0}$, we have partitions

$$\boldsymbol{M}(\boldsymbol{v}) = \left[\begin{array}{c|c} \boldsymbol{M}_{11}(\boldsymbol{v}) & \boldsymbol{M}_{12}(\boldsymbol{v}) \\ \hline \boldsymbol{M}_{21}(\boldsymbol{v}) & \boldsymbol{M}_{22}(\boldsymbol{v}) \end{array}\right]$$

$$\boldsymbol{M}^{-1}(\boldsymbol{v}) = \left[\begin{array}{c|c} \boldsymbol{M}^{11}(\boldsymbol{v}) & \boldsymbol{M}^{12}(\boldsymbol{v}) \\ \hline \boldsymbol{M}^{21}(\boldsymbol{v}) & \boldsymbol{M}^{22}(\boldsymbol{v}) \end{array}\right]$$

for $\boldsymbol{M}_{11}(\boldsymbol{v}), \boldsymbol{M}^{11}(\boldsymbol{v}) \in \mathbb{R}^{s\times s}$. Just as $\boldsymbol{M}^{-1}(\boldsymbol{v})$ approximates the covariance matrix of the estimator of $\boldsymbol{\theta}$, $\boldsymbol{M}^{11}(\boldsymbol{v})$ does so for the covariance matrix of the estimator of $\boldsymbol{\theta}_{[1]}$. Here

$$\boldsymbol{M}^{11} = \left(\boldsymbol{M}_{11} - \boldsymbol{M}_{12}\boldsymbol{M}_{22}^{-1}\boldsymbol{M}_{21}\right)^{-1}$$

We shall substitute

$$\left(\boldsymbol{M}^{11}\right)^{-1} = \boldsymbol{M}_{11} - \boldsymbol{M}_{12}\boldsymbol{M}_{22}^{-1}\boldsymbol{M}_{21} =: \boldsymbol{M}_{22} \mid \boldsymbol{M}$$

i.e., the Schur complement of $\boldsymbol{M}_{22}$ w.r.t. $\boldsymbol{M}$, for the information matrix associated with the parameters of interest.

Proposed formulation

Problem

Find $\boldsymbol{v} \in \mathcal{V}$ *to maximize*

$$P(\boldsymbol{v}) = \min_{k=1,\dots,s} \frac{1}{c_k} \Phi_{\mathsf{E}_k}\big(\boldsymbol{M}_{11}(\boldsymbol{v}) - \boldsymbol{M}_{12}(\boldsymbol{v})\boldsymbol{M}_{22}(\boldsymbol{v})^{-1}\boldsymbol{M}_{21}(\boldsymbol{v})\big)$$

subject to

$$\mathrm{eff}(\boldsymbol{v} \mid \Phi_D) \geq \eta$$

The maximum values of the Φ_{E_k}-optimum criteria

$$c_k = \max\Big\{\Phi_{\mathsf{E}_k}(\boldsymbol{M}_{11}(\boldsymbol{v}) - \boldsymbol{M}_{12}(\boldsymbol{v})\boldsymbol{M}_{22}(\boldsymbol{v})^{-1}\boldsymbol{M}_{21}(\boldsymbol{v})) : \boldsymbol{v} \in \mathcal{V},\ \mathrm{eff}(\boldsymbol{v} \mid \Phi_D) \geq \eta\Big\}$$

are to be computed and stored beforehand.

Rationale behind the constraint on D-efficiency

The approach is valid as long as $\det(\boldsymbol{M}(\boldsymbol{v})) > 0$, but under this constraint the set of feasible weights fails to be compact. We could replace it by $\det(\boldsymbol{M}(\boldsymbol{v})) > \epsilon$ for an arbitrarily small $\epsilon > 0$, but this would require clarification of what 'small' actually means.

This dilemma can be resolved by arbitrarily setting a minimal acceptable value η of D-efficiency:

$$\mathrm{eff}(\boldsymbol{v} \mid \Phi_D) = \frac{\det^{1/m}(\boldsymbol{M}(\boldsymbol{v}))}{\det^{1/m}(\boldsymbol{M}(\boldsymbol{v}^\star_{\mathsf{D}}))} \geq \eta$$

where $\boldsymbol{v}^\star_{\mathsf{D}}$ stands for a D-optimum design.

Reframed problem

Redefine

$$\mathcal{Q}_k = \{\boldsymbol{Q} \in \mathbb{R}^{s\times k} : \boldsymbol{Q}^\top \boldsymbol{Q} = \boldsymbol{I}_k\}$$

Using the *minimum principle* we get the following convenient formulation:

Problem

Find $\boldsymbol{v} \in \mathcal{V}$ *to maximize*

$$P(\boldsymbol{v}) = \min_{k=1,\dots,s} \frac{1}{c_k} \min_{\boldsymbol{Q}\in\mathcal{Q}_k} \operatorname{trace}\Big(\boldsymbol{Q}^\top \big(\boldsymbol{M}_{11}(\boldsymbol{v}) - \boldsymbol{M}_{12}(\boldsymbol{v})\boldsymbol{M}_{22}(\boldsymbol{v})^{-1}\boldsymbol{M}_{21}(\boldsymbol{v})\big)\boldsymbol{Q}\Big)$$

subject to

$$\operatorname{eff}(\boldsymbol{v} \mid \Phi_D) \geq \eta$$

Problem convexity

The optimality criterion is concave, which results from the following properties of the Schur complement:

- $\boldsymbol{M} \succ \boldsymbol{0} \implies \boldsymbol{M}_{22} \mid \boldsymbol{M} \succ \boldsymbol{0}$
- The mapping $\boldsymbol{M} \mapsto \boldsymbol{M}_{22} \mid \boldsymbol{M}$ is nondecreasing and matrix-concave over the cone of positive definite matrices.

Reduction to SIP

Problem

Find $(\boldsymbol{v}, \alpha) \in \mathcal{V} \times \mathbb{R}$ *so as to maximize* α *subject to the constraints*

$$\frac{1}{c_k} \operatorname{trace}\big(\boldsymbol{Q}_k^\top \big(\boldsymbol{M}_{11}(\boldsymbol{v}) - \boldsymbol{M}_{12}(\boldsymbol{v}) \boldsymbol{M}_{22}(\boldsymbol{v})^{-1} \boldsymbol{M}_{21}(\boldsymbol{v})\big) \boldsymbol{Q}_k\big) \geq \alpha,$$
$$\forall \boldsymbol{Q}_k \in \mathcal{Q}_k, \quad k = 1, \dots, s$$

$$\det{}^{1/m}(\boldsymbol{M}(\boldsymbol{v})) \geq b$$

Here $b = \eta \det{}^{1/m}(\boldsymbol{M}(\boldsymbol{v}_{\mathsf{D}}^\star))$ is a constant.

This problem can also be solved using the method of outer approximations. It alternates between the eigen-decomposition of the current FIM and solving a finite convex minimax problem with respect to the design weights $\boldsymbol{v}$.

Eigen-decomposition step

Given $\boldsymbol{v} \in \mathcal{V}$, determine the eigenvalues λ_ℓ, $\ell = 1, \dots, s$ of $\boldsymbol{M}_{11}(\boldsymbol{v}) - \boldsymbol{M}_{12}(\boldsymbol{v})\boldsymbol{M}_{22}(\boldsymbol{v})^{-1}\boldsymbol{M}_{21}(\boldsymbol{v})$ with the corresponding orthonormal eigenvectors $\boldsymbol{q}_1, \dots, \boldsymbol{q}_s$. Then compute

$$k^\star = \arg \min_{k=1,\dots,s} \frac{1}{c_k} \sum_{\ell=1}^{k} \lambda_\ell,$$

$$P(\boldsymbol{v}) = \frac{1}{c_{k^\star}} \sum_{\ell=1}^{k^\star} \lambda_\ell,$$

and form the matrix $\boldsymbol{Q}^\star(\boldsymbol{v}) = \left[\ \boldsymbol{q}_1 \ \middle|\ \dots \ \middle|\ \boldsymbol{q}_{k^\star}\ \right] \in \mathbb{R}^{s \times k^\star}$.

Finite maximin step

Given a finite set $\mathcal{Q}$ with elements being some orthogonal $s \times k$ matrices (s is fixed, while $k \in \{1, \ldots, s\}$ may vary from matrix to matrix), find $(\boldsymbol{v}^\star, \alpha^\star) \in \mathcal{V} \times \mathbb{R}$ that maximizes α subject to

$$\frac{1}{c_{\mathsf{cols}(\boldsymbol{Q})}} \operatorname{trace}(\boldsymbol{Q}^\top \big(\boldsymbol{M}_{11}(\boldsymbol{v}) - \boldsymbol{M}_{12}(\boldsymbol{v})\boldsymbol{M}_{22}(\boldsymbol{v})^{-1}\boldsymbol{M}_{21}(\boldsymbol{v})\big)\boldsymbol{Q}) \geq \alpha, \quad \forall \boldsymbol{Q} \in \mathcal{Q}.$$

$$\det^{1/m}(\boldsymbol{M}(\boldsymbol{v})) \geq b$$

The polyhedral form of the set $\mathcal{V}$ and the convexity of the constraints make it possible to drastically reduce the problem dimensionality through the use of *generalized simplicial decomposition*, see (Bertsekas and Yu, 2011; Uciński, 2022) for more details.

Simulation example

Consider diffusive transport of the contaminant in a spatial domain $\Omega = [0,1]^2$ over a time horizon $T = (0,1]$, which is modelled by the advection-diffusion equation

$$\frac{\partial y}{\partial t} + \boldsymbol{\beta} \cdot \nabla y - \nabla \cdot (a \nabla y) = u$$

where

- $y = y(\boldsymbol{x}, t)$ – contaminant concentration at spatial point $\boldsymbol{x} \in \Omega$ and time instant $t \in T$
- $\boldsymbol{\beta} = (-0.1, 0.7)$ – known wind velocity
- ∇ – the spatial gradient
- $u(\boldsymbol{x}, t)$ – forcing term modelling the contamination source
- $a(\boldsymbol{x})$ – diffusion coefficient.

This equation constitutes a physical model of the mesoscale atmospheric motion.

Boundary and initial conditions

The PDE is complemented with the boundary conditions

$$\begin{aligned} y &= 0 \quad \text{on } \partial\Omega^- \times T, \\ \frac{\partial y}{\partial \boldsymbol{n}} &= 0 \quad \text{on } \partial\Omega^+ \times T, \end{aligned}$$

and the initial condition

$$y\big|_{t=0} = 0 \quad \text{in } \Omega$$

where $\partial\Omega$ is the boundary of Ω,
$\partial\Omega^- = \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{v} \cdot \boldsymbol{n} < 0\} = [0,1] \times \{0\} \cup \{1\} \times [0,1]$,
$\partial\Omega^+ = \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{v} \cdot \boldsymbol{n} \geq 0\} = \{0\} \times [0,1] \cup [0,1] \times \{1\}$,
and $\partial y / \partial \boldsymbol{n}$ stands for the derivative of y in the direction of the outward normal of $\partial\Omega$, $\boldsymbol{n}$.

Mobile contamination source

It is known that the contamination source is mobile, with emission intensity b, initially located at an unknown spatial point $\boldsymbol{z} = (z_1, z_2)$ and moving in uniform motion with constant velocity $\boldsymbol{v}$ parallel to the x_1-axis, i.e., $\boldsymbol{v} = (w, 0)$ with unknown w. Mathematically, the source is modelled as a slender Gaussian function

$$u(\boldsymbol{x}, t) = b \exp(-50\|\boldsymbol{x} - (\boldsymbol{z} + \boldsymbol{v}t)\|)$$

which emulates the action of a mobile pointwise source.

Diffusion coefficient

The diffusion coefficient is modelled as a linear function

$$a(\boldsymbol{x}) = a_0 + a_1 x_1 + a_2 x_2$$

where the values of a_0, a_1 and a_2 are unknown.
All the unknown coefficients are collected in the vector

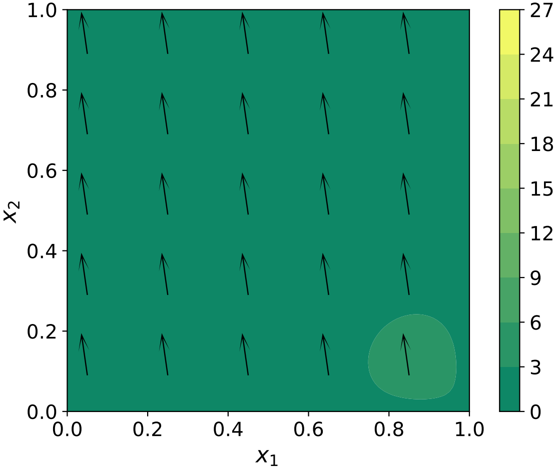
$$\boldsymbol{\theta} = (a_0, a_1, a_2, b, z_1, z_2, w)$$

For simulation purposes, the following nominal value of this vector was assumed:

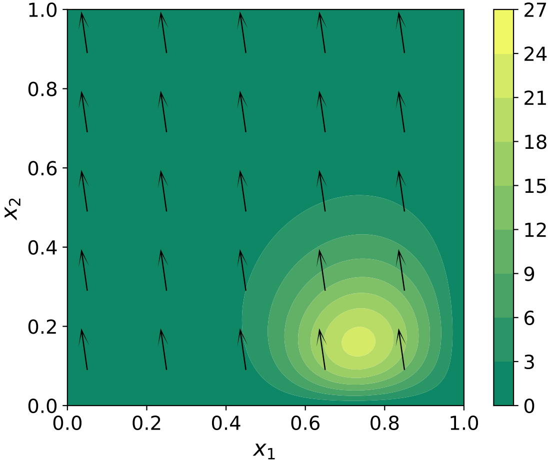
$$\boldsymbol{\theta}^0 = (0.1, -0.05, 0.15, 330, 0.9, 0.1, -0.8)$$

The parameter vector $\boldsymbol{\theta}$ is going to be estimated using $n = 150$ sensors selected from a total of 780 sensor network nodes residing at some nonnegligible distance from the trajectory of the moving source. We also set $\eta = 0.1$.

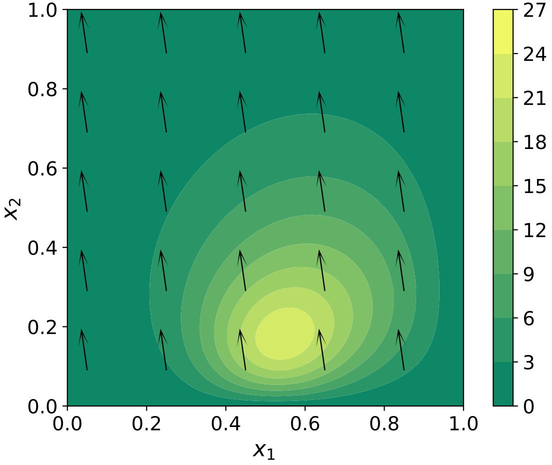
Evolution of the contaminant concentration



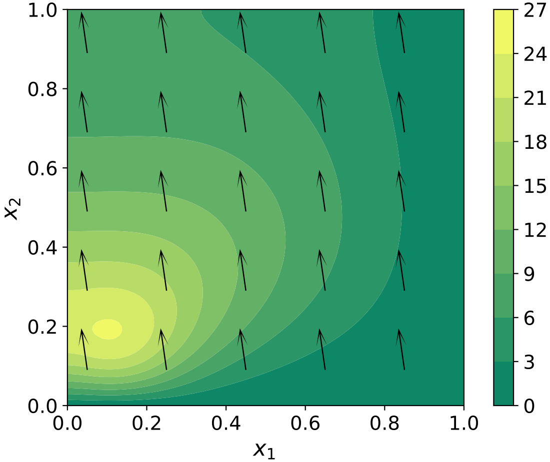
Evolution of the contaminant concentration



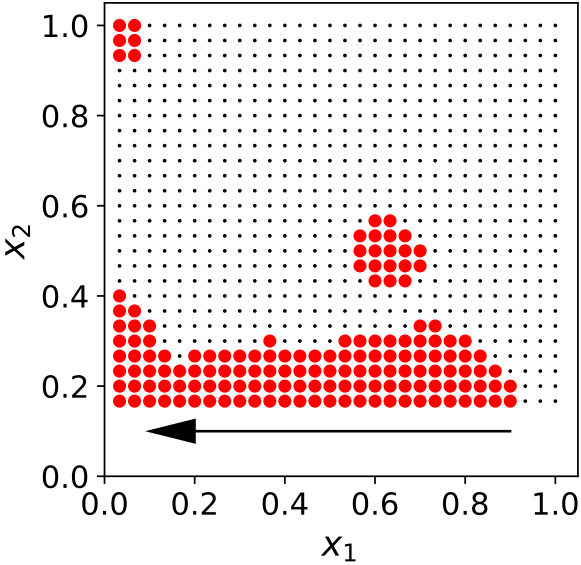
Evolution of the contaminant concentration



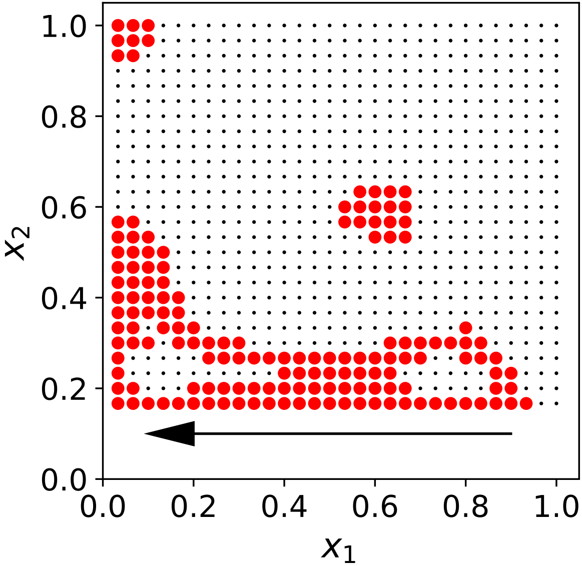
Evolution of the contaminant concentration



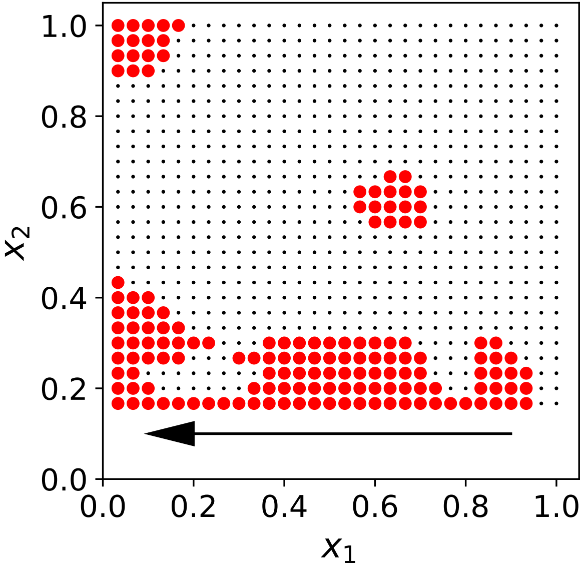
D-optimum design



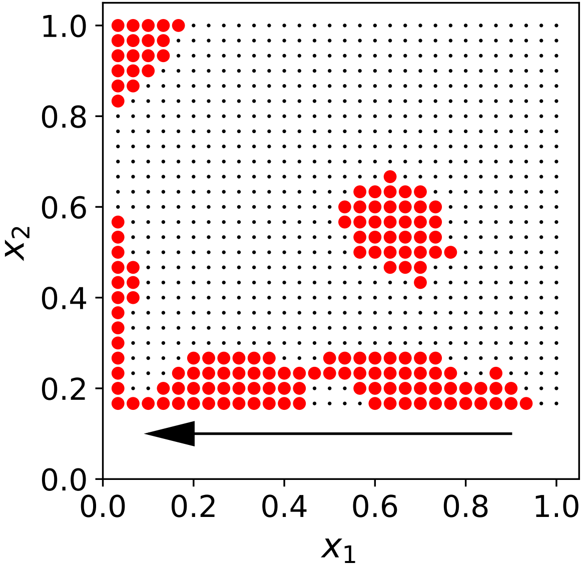
POI: b, z_1, z_2, w, MMEff $= 0.783$



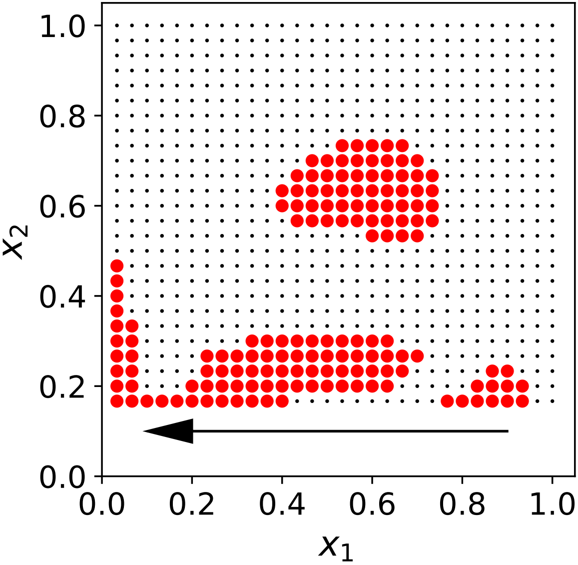
POI: z_1, z_2, MMEff $= 0.881$



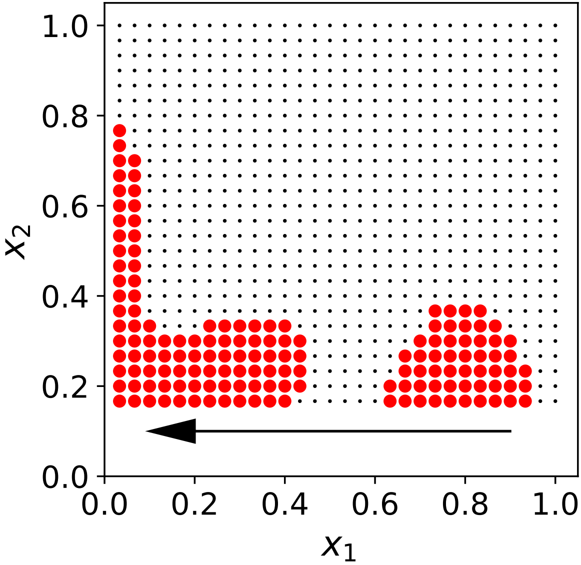
POI: a_0, a_1, a_2, MMEff $= 0.818$



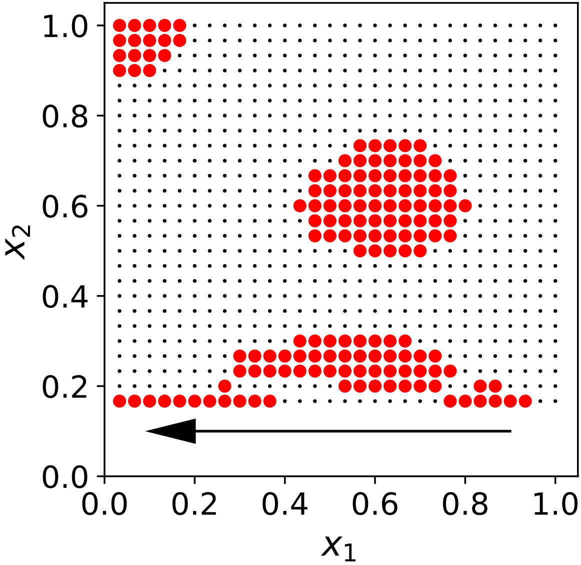
POI: a_0



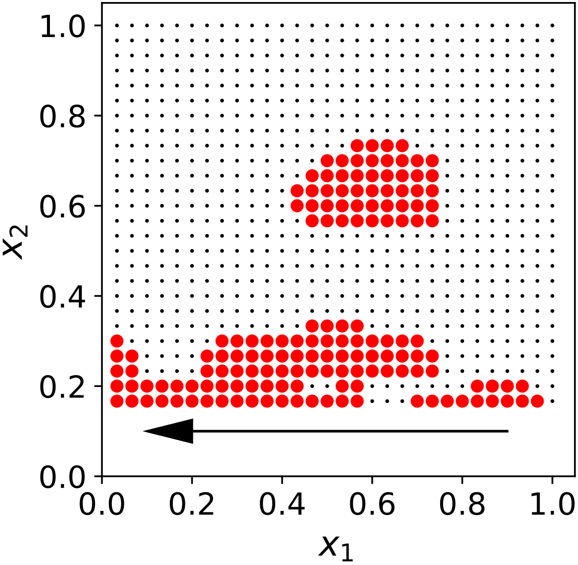
POI: a_1



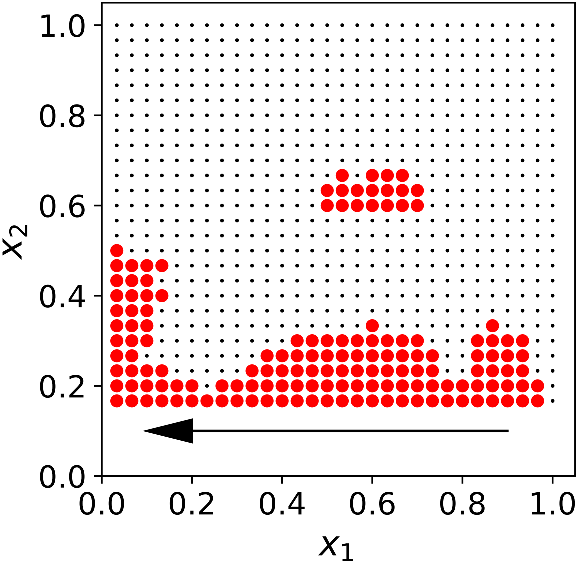
POI: a_2



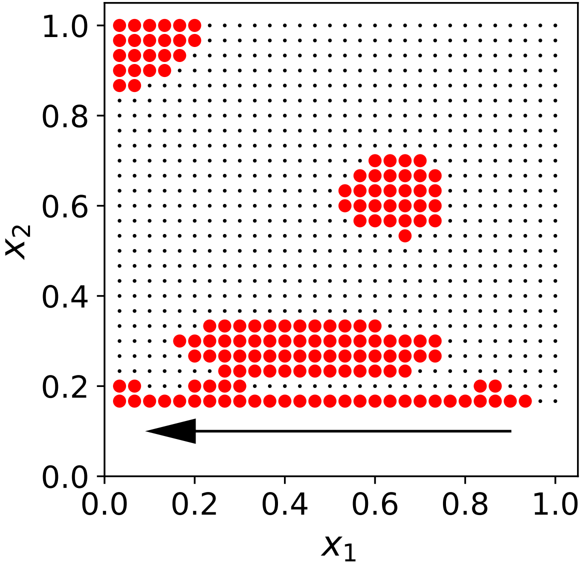
POI: b



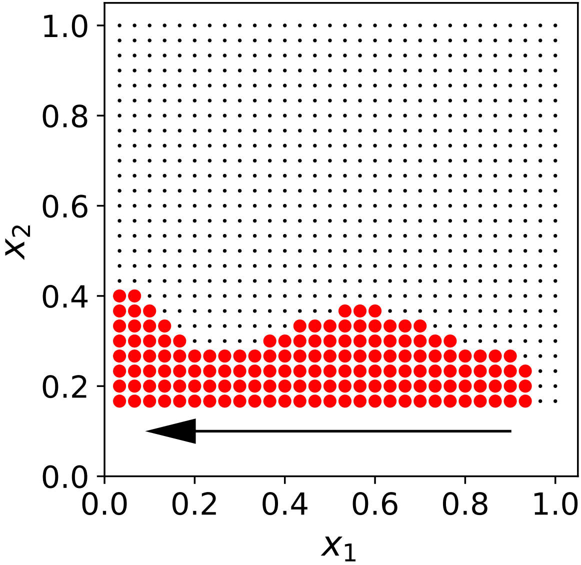
POI: z_1



POI: z_2



POI: w



Concluding remarks

1. Computation of $\mathbb{O}_s$-maximin efficient designs is much more complicated than the same task for the whole set of parameters, since the finite maximin problems in the relaxation procedure are no longer linear programming ones. Nevertheless, it is still relatively easy to implement due to the problem convexity, and extremely fast even for very large sets of candidate sites (due to application of generalized simplicial decomposition).
2. In spite of the conservatism of the adopted optimality criterion, the produced optimized lower bounds to the efficiencies of the conceivable criteria can be quite high, i.e., the produced sensor locations can be very close to optimal locations for common alphabetic criteria.
3. The method is going to bo combined with the branch-and-bound procedure to compute optimal discrete designs.

References

- D.P. Bertsekas and H. Yu, "A unifying polyhedral approximation framework for convex optimization, " *SIAM Journal on Optimization*, vol. 21, No. 1, pp. 333–360, 2011
- K. Burclová and A. Pázman, "Optimal design of experiments via linear programming," *Statistical Papers*, vol. 57, pp. 893–910, 2016.
- R. Harman, "Minimal efficiency of designs under the class of orthogonally invariant information criteria," *Metrika*, vol. 60, no. 2, pp. 137–153, 2004.
- D. Uciński and M. Patan, "Maximin Efficient Sensor Location for Parameter Estimation of Spatiotemporal Systems," *60th IEEE Conference on Decision and Control (CDC)*, Austin, TX, USA, pp. 2601–2606, 2021.
- D. Uciński, "E-optimum sensor selection for estimation of subsets of parameters," *Measurement*, vol. 187, paper no. 110286, 2022.